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On Dual Formulations of Massive Tensor Fields

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Abstract

In this paper we investigate dual formulations for massive tensor fields. Usual procedure for construction of such dual formulations based on the use of first order parent Lagrangians in many cases turns out to be ambiguous. We propose to solve such ambiguity by using gauge invariant description of massive fields which works both in Minkowski space as well as (Anti) de Sitter spaces. We illustrate our method by two concrete examples: spin-2 "tetrad" field ${h_\mu}^a$, the dual field being "Lorentz connection" ${\omega_\mu}^{ab}$ and "Riemann" tensor ${R_{\mu\nu}}^{ab}$ with the dual ${\Sigma_{\mu\nu}}^{abc}$.

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Introduction

Investigations of dual formulations for tensor fields are important for understanding of alternative formulations of known theories like gravity as well as understanding of their role in superstrings. Common procedure for obtaining such dual formulations is based on the parent first order Lagrangians. But such procedure turns out to be simple and unambiguous for completely antisymmetric form fields only. Dual formulations for more general massless fields where investigated recently in [1, 2, 3]. To illustrate the reason let us compare two simplest cases: massive $s = 1$ and $s = 2$ fields. First order formulation of massive spin-1 field requires two fields $(F^{[\mu\nu]}, A_\mu)$ treated as independent ones. The first order Lagrangian has the form:

$$\mathcal{L}_I = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{m^2}{2}{A_\mu}^2$$

If one solves the algebraic equation of motion for the $F^{\mu\nu}$ field and put result back into the Lagrangian one obtains usual second order description for massive spin-1 particle in terms of vector field A_μ. But then the mass $m \neq 0$ the equation of motion for vector field

$$\frac{\delta\mathcal{L}}{\delta A_\mu} = m^2 A_\mu + (\partial F)_\mu = 0$$

turns out to be algebraic too, so one can proceed by solving this equation. Then putting the result back into the Lagrangian one gets dual formulation of the same particle in terms of antisymmetric tensor field $F^{\mu\nu}$:

$$\mathcal{L}_{II} = -\frac{1}{2}{(\partial F)_\mu}^2 + \frac{m^2}{4}{F_{\mu\nu}}^2$$

where in order to get canonical normalization of kinetic term we make a rescaling $F^{\mu\nu} \to mF^{\mu\nu}$. Note that the kinetic term is invariant under the gauge transformations

$$\delta F^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}\partial_\alpha\xi_\beta$$

As a result in the massless limit $m \to 0$ such theory describes massless spin-0 particle.

Now let us turn to the spin-2 case. The simplest and most common description of such particle uses symmetric second rank tensor field $h_{(\mu\nu)}$. But there is no any combination of first derivatives for this field which will be invariant under the gauge transformations $\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. So to construct first order Lagrangian one has to abandon the symmetry property and use "tetrad" field ${h_\mu}^a$ with modified gauge transformations $\delta {h_\mu}^a = \partial_\mu\xi^a$. Then introducing auxiliary field ${\omega_\mu}^{[ab]}$ one can construct the following first order Lagrangian:

$$\begin{aligned}\mathcal{L}_I &= -\frac{1}{2}\omega^{\mu,\alpha\beta}\omega_{\alpha,\mu\beta} + \frac{1}{2}\omega^\mu\omega_\mu - \partial_\beta\omega^{\mu,\alpha\beta}h_{\alpha\mu} - \partial^\alpha\omega^\mu h_{\mu\alpha} + (\partial\omega)h - \\ &\quad -\frac{m^2}{2}(h^{(\mu\nu)}h_{(\mu\nu)} - h^2) + ah^{[\mu\nu]}h_{[\mu\nu]}\end{aligned}$$

Now we face the ambiguity in the mass terms. Indeed, if we solve the algebraic equation of motion for the ω field and put result back into the Lagrangian we obtain usual second order Lagrangian for the symmetric part of ${h_\mu}^a$ while antisymmetric part turns out to be

non dynamical and completely decouples from it. But it is the mass terms that determines the structure of kinetic terms in the dual theory so that starting from different values for a parameter we will get different dual versions. For example, if one drops antisymmetric part from the mass terms by choosing $a = 0$ it will persist in the kinetic terms serving as a Lagrangian multiplier and giving differential constraint on ω field. If one will not insist on the canonical form of the massless part in the first order parent Lagrangian then even more arbitrary parameters could be introduced [4, 5, 6].

In this paper we are going to show that one of the way to resolve such ambiguities is to use gauge invariant description of massive tensor fields [7, 8, 9, 10]. To illustrate how such a procedure works, let us ones again use simplest spin-1 case. As is well known gauge invariant description of massive spin-1 particle requires introduction of additional Goldstone scalar field. Working with first order formalism it is natural to use first order description for all fields under consideration. So we introduce a pair (π^μ, φ) and consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{4} {F_{\mu\nu}}^2 - \frac{1}{2} F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{2} {\pi_\mu}^2 + \pi^\mu \partial_\mu \varphi - m \pi^\mu A_\mu \qquad (1)$$

It is easy to check that this Lagrangian is invariant under the gauge transformations:

$$\delta A_\mu = \partial_\mu \lambda \qquad \delta\varphi = m\lambda$$

Moreover, if one solves algebraic equation of motion for $F^{\mu\nu}$ and π^μ fields and put the result back into the Lagrangian one obtains usual second order formulation of massive spin-1 particle in terms of vector A_μ and scalar φ fields. Now let us try to solve the equations for the A_μ and φ fields instead. First of all we face the fact that vector field A_μ enters the Lagrangian only linearly working as a Lagrangian multiplier. Thus it's equation

$$\frac{\delta\mathcal{L}}{\delta A^\mu} = (\partial F)_\mu - m\pi_\mu = 0$$

could be solved for the π^μ field and not for the field A_μ itself. Moreover the equation for the scalar field φ

$$\frac{\delta\mathcal{L}}{\delta\varphi} = -(\partial\pi) = 0$$

is not an independent one. Indeed it is easy to check that two equations satisfy the relation:

$$\partial^\mu \frac{\delta\mathcal{L}}{\delta A^\mu} - m\frac{\delta\mathcal{L}}{\delta\varphi} = 0$$

which is just a consequence of the invariance under λ gauge transformations. So we can't express φ field in terms of the others, but it's not necessary because if put the solution of A_μ equation $\pi_\mu = -\frac{1}{m}(\partial F)_\mu$ into the initial Lagrangian the scalar field φ completely decouples leaving us with:

$$\mathcal{L} = -\frac{1}{2m^2} {(\partial F)_\mu}^2 + \frac{1}{4} {F_{\mu\nu}}^2$$

In the following two sections we give two explicit examples of the dual formulations for massive tensor fields obtained by the procedure described above. The first one will be for the massive "tetrad" field ${h_\mu}^a$ the dual field being "Lorentz connection" ${\omega_\mu}^{ab}$. The second example will be the "Riemann" tensor ${R_{\mu\nu}}^{ab}$ with the dual ${\Sigma_{\mu\nu}}^{abc}$.

1 Massive spin 2

Gauge invariant description of massive spin-2 particle requires introduction of two Goldstone fields, namely the vector A_μ and scalar φ ones [7]. As we have already noted, it is natural to use first order formulation for all fields under the consideration, so we introduce three pairs of fields: $(h_\mu{}^a, \omega_\mu{}^{ab})$, (A_μ, F^{ab}) and (φ, π^a) [10]. Our starting point will be the sum of free massless Lagrangians in flat Minkowski space:

$$\begin{aligned}
\mathcal{L}_0 &= \mathcal{L}_0(\omega_\mu{}^{ab}, h_\mu{}^a) + \mathcal{L}_0(F^{ab}, A_\mu) + \mathcal{L}_0(\pi^a, \varphi) \qquad (2)\\
\mathcal{L}_0(\omega_\mu{}^{ab}, h_\mu{}^a) &= \frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^{ac}\omega_\nu{}^{bc} - \frac{1}{2}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} \omega_\mu{}^{ab}\partial_\nu h_\alpha{}^c \\
\mathcal{L}_0(F^{ab}, A_\mu) &= \frac{1}{4}F_{ab}{}^2 - \frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\} F^{ab}\partial_\mu A_\nu \\
\mathcal{L}_0(\pi^a, \varphi) &= -\frac{1}{2}\pi_a{}^2 + \left\{ {}^{\mu}_{a} \right\} \pi^a\partial_\mu\varphi
\end{aligned}$$

Here

$$\left\{ {}^{\mu\nu}_{ab} \right\} = \delta_a{}^\mu\delta_b{}^\nu - \delta_a{}^\nu\delta_b{}^\mu$$

and so on. This Lagrangian is invariant under the following local gauge transformations:

$$\delta_0 h_{\mu a} = \partial_\mu\xi_a + \eta_{\mu a} \qquad \delta_0\omega_\mu{}^{ab} = \partial_\mu\eta^{ab} \qquad \delta_0 A_\mu = \partial_\mu\lambda \qquad (3)$$

In (Anti) de Sitter spaces with nonzero cosmological constant gauge invariance requires introduction of mass-like terms into the Lagrangian as well as appropriate corrections for the gauge transformation laws in much the same way as nonzero mass in Minkowski space does. One of the pleasant features of gauge invariant description of massive particles is the possibility to consider general case of massive particle in (Anti) de Sitter as well as flat spaces including all possible massless and partially massless limits [7, 8]. Moreover, as was shown in [2], duality procedure for the massless fields in spaces with nonzero cosmological constant works very similar to the one for massive fields in flat space. Here we consider such a general case with nonzero mass and cosmological constant. To simplify the formula we restrict ourselves to space-times with the dimension $d = 4$ only, but the whole construction could be easily generalized to any dimensions $d \geq 3$.

Working with the first order formalism it is very convenient to use tetrad formulation of the underlying (Anti) de Sitter space. We denote tetrad as $e_\mu{}^a$ (let us stress that it is not a dynamical quantity here, just a background field) and Lorentz covariant derivative as D_μ. (Anti) de Sitter space is a constant curvature space with zero torsion, so we have:

$$D_{[\mu}e_{\nu]}{}^a = 0, \qquad [D_\mu, D_\nu]v^a = \kappa(e_\mu{}^a e_\nu{}^b - e_\mu{}^b e_\nu{}^a)v_b \qquad (4)$$

where $\kappa = -2\Lambda/(d-1)(d-2) = -\Lambda/3$.

Now we replace all the derivatives in the Lagrangian and gauge transformation laws by the covariant ones. Due to noncommutativity of covariant derivatives the Lagrangian becomes non invariant and we get:

$$\delta_0\mathcal{L}_0 = 2\kappa\omega^a\xi_a - 2\kappa h^{ab}\eta_{ab} \qquad (5)$$

But the invariance could be restored by adding low derivative terms to the Lagrangian:

$$\begin{aligned}\Delta\mathcal{L} &= \frac{m}{\sqrt{2}}[2\omega^\mu A_\mu + F^{\mu\nu}h_{\mu\nu}] + a_0\pi^\mu A_\mu - \\ &\quad - \frac{{a_0}^2}{6}(h^{\mu\nu}h_{\nu\mu} - h^2) + \frac{ma_0}{\sqrt{3}}h\varphi + m^2\varphi^2 \qquad (6)\end{aligned}$$

as well as corresponding additional terms to the gauge transformation laws:

$$\begin{aligned}\delta' h_\mu{}^a &= \frac{m}{\sqrt{2}}e_\mu{}^a\lambda \qquad \delta'\omega_\mu{}^{ab} = -\frac{{a_0}^2}{6}(e_\mu{}^a\xi^b - e_\mu{}^b\xi^a) \\ \delta' F^{ab} &= -m\sqrt{2}\eta^{ab} \qquad \delta' A_\mu = \frac{m}{\sqrt{2}}\xi_\mu \qquad (7) \\ \delta'\pi^a &= \frac{ma_0}{\sqrt{2}}\xi^a \qquad \delta'\varphi = -a_0\lambda \qquad (8)\end{aligned}$$

Here ${a_0}^2 = 3m^2 + 6\kappa$. The case $a_0 = 0$ corresponds to the so called partial massless particles in the de Sitter space and requires special treatment. In what follows we will assume that $a_0 \neq 0$.

Now having in our disposal complete first order Lagrangian we try to solve the equations for the $h_\mu{}^a$, A_μ and φ fields. Let us start with the equation for A_μ field. Inspection of the Lagrangian reveals that this field enters the Lagrangian only linearly, so that it's equation

$$\frac{\delta\mathcal{L}}{\delta A_\mu} = D^\alpha F_{\alpha\mu} + m\sqrt{2}\omega_\mu + a_0\pi_\mu \qquad (9)$$

could not be solved for the A_μ field itself. But for $a_0 \neq 0$ it could be easily solved for π^μ:

$$\pi_\mu = -\frac{1}{a_0}[D^\alpha F_{\alpha\mu} + m\sqrt{2}\omega_\mu] \qquad (10)$$

In this, if one put this result back into the Lagrangian, both A_μ as well as π_μ drop out. Two other equations look like:

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta h^{\mu\nu}} &= R_{\nu\mu} - \frac{1}{6}g_{\mu\nu}R + \frac{m}{\sqrt{2}}F_{\mu\nu} - \frac{{a_0}^2}{3}(h_{\nu\mu} - g_{\mu\nu}h) + \frac{ma_0}{\sqrt{2}}g_{\mu\nu}\varphi = 0 \\ \frac{\delta\mathcal{L}}{\delta\varphi} &= -(D\pi) + \frac{ma_0}{\sqrt{2}}h + 2m^2\varphi = 0 \qquad (11)\end{aligned}$$

Here we introduce the tensor

$$R_{\mu\nu}{}^{\alpha\beta} = D_\mu\omega_\nu{}^{\alpha\beta} - D_\nu\omega_\mu{}^{\alpha\beta}$$

as well as it's contractions $R_\mu{}^\alpha = R_{\mu\nu}{}^{\alpha\nu}$ and $R = R_\mu{}^\mu$. Note that the derivatives D_μ are covariant under the background Lorentz connection only, so $R_{\mu\nu}{}^{\alpha\beta}$ is not truly gauge invariant object. For example, under $\delta\omega_\mu{}^{\alpha\beta} = D_\mu\eta^{\alpha\beta}$ we have $\delta R_{\mu\nu} = -2\kappa\eta_{\mu\nu}$. By taking a trace of the $h_{\mu\nu}$ equation

$$g^{\mu\nu}\frac{\delta\mathcal{L}}{\delta h^{\mu\nu}} = -R + {a_0}^2 h + \frac{4ma_0}{\sqrt{3}}\varphi = 0$$

and comparing it with the φ equation it is easy to see that this equations are not independent but satisfy:

$$D^\mu \frac{\delta \mathcal{L}}{\delta A^\mu} + \frac{m}{\sqrt{2}} g^{\mu\nu} \frac{\delta \mathcal{L}}{\delta h^{\mu\nu}} - a_0 \frac{\delta \mathcal{L}}{\delta \varphi} = 0$$

This last relation clearly shows that this dependency is just the consequence of the λ gauge invariance. So we cannot solve this equations for the $h_{\mu\nu}$ and φ fields simultaneously. But if one solves the $h_{\mu\nu}$ equation as:

$$h_{\mu\nu} = \frac{3}{{a_0}^2}[R_{\nu\mu} - \frac{1}{6} g_{\mu\nu} R - \frac{m}{\sqrt{2}} F_{\mu\nu} - \frac{m a_0}{3\sqrt{2}} g_{\mu\nu}\varphi] \tag{12}$$

and put this expression into original Lagrangian (together with expression for π_μ) one sees that the scalar field φ completely decouples. As a final result one obtains the second order Lagrangian containing two fields only: the "gauge" field ${\omega_\mu}^{\alpha\beta}$ and "Goldstone" field $F^{\alpha\beta}$:

$$\begin{aligned}
\mathcal{L}_{II} &= \frac{1}{2}(R^{\mu\nu} R_{\nu\mu} - \frac{1}{3} R^2) - \frac{1}{6}(D^\alpha F_{\alpha\mu})^2 + \\
&+ \frac{m}{\sqrt{2}}(\omega^{\mu,\nu\alpha} D_\alpha F_{\mu\nu} + \frac{1}{3}\omega^\mu D^\alpha F_{\alpha\mu}) - \\
&-(\frac{m^2}{2} + \kappa)\omega^{\mu,\alpha\beta}\omega_{\alpha,\mu\beta} + (\frac{m^2}{6} + \kappa)\omega^\mu \omega_\mu + \frac{\kappa}{2}{F_{\mu\nu}}^2
\end{aligned} \tag{13}$$

This Lagrangian has general structure common for all gauge invariant Lagrangians describing massive particles. It contains sum of the kinetic terms for two fields ${\omega_\mu}^{\alpha\beta}$ and $F^{\alpha\beta}$, cross terms with one derivative as well as mass terms. (It could seems strange to have mass-like term for the Goldstone field $F^{\mu\nu}$, but it's not unnatural in (Anti) de Sitter spaces.) And indeed this Lagrangian is invariant under the following gauge transformations:

$$\delta {\omega_\mu}^{\alpha\beta} = D_\mu \eta^{\alpha\beta} \qquad \delta F^{\alpha\beta} = -m\sqrt{2}\eta^{\alpha\beta} \tag{14}$$

Besides, as a remnant of the ξ-symmetry of the initial first order Lagrangian, the second order Lagrangian is invariant also under the local shifts:

$$\delta {\omega_\mu}^{\alpha\beta} = -{e_\mu}^\alpha \xi^\beta + {e_\mu}^\beta \xi^\alpha \tag{15}$$

This invariance could be easily checked if one uses that under such transformations $\delta R_{\mu\nu} = 2D_\mu \xi_\nu + g_{\mu\nu}(D\xi)$ and take into account useful identity: $D^\nu R_{\mu\nu} - \frac{1}{2} D_\mu R = -2\kappa\omega_\mu$.

2 Massive ${R_{[\mu\nu]}}^{[ab]}$ tensor

For gauge invariant description of appropriate massive particle one needs two additional Goldstone fields: ${\Phi_{\mu\nu}}^a$ and ${h_\mu}^a$ [8]. So to construct first order form of such description we introduce three pairs of fields [9, 10]: $({\Sigma_{\mu\nu}}^{abc}, {R_{\mu\nu}}^{ab})$, $({\Omega_\mu}^{abc}, {\Phi_{\mu\nu}}^a)$ and $({\omega_\mu}^{ab}, {h_\mu}^a)$. The sum of flat space massless Lagrangians:

$$\begin{aligned}
\mathcal{L}_0 &= \mathcal{L}_0({\Sigma_{\mu\nu}}^{abc}, {R_{\mu\nu}}^{ab}) + \mathcal{L}_0({\Omega_\mu}^{abc}, {\Phi_{\mu\nu}}^a) + \mathcal{L}_0({\omega_\mu}^{ab}, {h_\mu}^a) \\
\mathcal{L}_0({\Sigma_{\mu\nu}}^{abc}, {R_{\mu\nu}}^{ab}) &= -\frac{3}{8}\left\{ {}^{\mu\nu\alpha\beta}_{abcd} \right\} {\Sigma_{\mu\nu}}^{abe} {\Sigma_{\alpha\beta}}^{cde} + \frac{1}{4}\left\{ {}^{\mu\nu\alpha\beta\gamma}_{abcde} \right\} {\Sigma_{\mu\nu}}^{abc} \partial_\alpha {R_{\beta\gamma}}^{de} \\
\mathcal{L}_0({\Omega_\mu}^{abc}, {\Phi_{\mu\nu}}^a) &= -\frac{3}{4}\left\{ {}^{\mu\nu}_{ab} \right\} {\Omega_\mu}^{acd} {\Omega_\nu}^{bcd} + \frac{1}{4}\left\{ {}^{\mu\nu\alpha\beta}_{abcd} \right\} {\Omega_\mu}^{abc} \partial_\nu {\Phi_{\alpha\beta}}^d
\end{aligned} \tag{16}$$

where $\mathcal{L}_0(\omega_\mu{}^{ab}, h_\mu{}^a)$ is the same as before is invariant under the following gauge transformations:

$$
\begin{aligned}
\delta_0 R_{\mu\nu}{}^{ab} &= \partial_\mu \chi_\nu{}^{ab} - \partial_\nu \chi_\mu{}^{ab} + \psi_{\mu,\nu}{}^{ab} - \psi_{\nu,\mu}{}^{ab} \qquad \delta_0 \Sigma_{\mu\nu}{}^{abc} = \partial_\mu \psi_\nu{}^{abc} - \partial_\nu \psi_\mu{}^{abc} \\
\delta_0 \Phi_{\mu\nu}{}^a &= \partial_\mu z_\nu{}^a - \partial_\nu z_\mu{}^a + \eta_{\mu\nu}{}^a \qquad \delta_0 \Omega_\mu{}^{abc} = \partial_\mu \eta^{abc} \\
\delta_0 h_{\mu a} &= \partial_\mu \xi_a + \eta_{\mu a} \qquad \delta_0 \omega_\mu{}^{ab} = \partial_\mu \eta^{ab}
\end{aligned}
\tag{17}
$$

Then we proceed in the same way as in the previous example. Again to simplify formula we restrict ourselves to spaces with dimension $d = 5$ only (it's the minimal dimension where tensor $R_{\mu\nu}{}^{ab}$ has physical degrees of freedom), but all the results could be generalized to arbitrary $d \geq 5$ case. First of all we go to (Anti) de Sitter space by changing all the derivatives by covariant ones. Further we compensate resulting noninvariance of the Lagrangian by adding additional low derivative terms to the Lagrangian:

$$
\begin{aligned}
\mathcal{L}_1 &= \frac{m}{2} \left\{ {}^{\mu\nu\alpha\beta}_{abcd} \right\} \Sigma_{\mu\nu}{}^{abc} \Phi_{\alpha\beta}{}^d - \frac{3m}{2} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} R_{\mu\nu}{}^{ad} \Omega_\alpha{}^{bcd} + \\
&\quad + \frac{a_0}{2} \left\{ {}^{\mu\nu}_{ab} \right\} \Omega_\mu{}^{abc} h_\nu{}^c + \frac{a_0}{2} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} \omega_\mu{}^{ab} \Phi_{\nu\alpha}{}^c - \\
&\quad - \frac{a_0{}^2}{8} \left\{ {}^{\mu\nu\alpha\beta}_{abcd} \right\} R_{\mu\nu}{}^{ab} R_{\alpha\beta}{}^{cd} - m a_0 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} R^{ab}_{\mu\nu} h_\alpha{}^c - 3m^2 \left\{ {}^{\mu\nu}_{ab} \right\} h_\mu{}^a h_\nu{}^b
\end{aligned}
\tag{18}
$$

as well as corresponding terms to the gauge transformations:

$$
\begin{aligned}
\delta_1 R_{\mu\nu}{}^{ab} &= -\frac{m}{4} e_{[\mu}{}^{[a} z_{\nu]}{}^{b]} \qquad \delta_1 h_\mu{}^a = 2a_0 z_\mu{}^a \\
\delta_1 \Sigma_{\mu\nu}{}^{abc} &= \frac{m}{8} e_{[\mu}{}^{[a} \eta_{\nu]}{}^{bc]} + \frac{a_0{}^2}{3} e_{[\mu}{}^{[a} \chi_{\nu]}{}^{bc]} - \frac{m a_0}{3} e_{[\mu}{}^{[a} e_{\nu]}{}^b \xi^{c]} \\
\delta_1 \Phi_{\mu\nu}{}^a &= 2m(\chi_{\mu,\nu}{}^a - \chi_{\nu,\mu}{}^a) + \frac{a_0}{6} e_{[\mu}{}^a \xi_{\nu]} \\
\delta_1 \Omega_\mu{}^{abc} &= -4m \psi_\mu{}^{abc} + \frac{a_0}{3} e_\mu{}^{[a} \eta^{bc]} \\
\delta_1 \omega_\mu{}^{ab} &= -a_0 \eta_\mu{}^{ab} + \frac{a_0{}^2}{3} \chi_\mu{}^{ab} - \frac{m a_0}{3} e_\mu{}^{[a} \xi^{b]}
\end{aligned}
\tag{19}
$$

Here $a_0{}^2 = 6m^2 + 3\kappa$. As in the previous case $a_0 = 0$ corresponds to partially massless particle which require special treatment. In what follows we will assume that $a_0 \neq 0$.

It will be convenient to introduce Lorentz covariant "field strength"

$$
\Sigma_{\mu\nu\alpha}{}^{abc} = D_{[\mu} \Sigma_{\nu\alpha]}{}^{abc}
$$

Because it's covariant under the background Lorentz connection only it's not fully gauge invariant[1]:

$$
\delta \Sigma_{\mu\nu}{}^{abc} = D_{[\mu} \psi_{\nu]}{}^{abc} \quad \Longrightarrow \quad \delta \Sigma_{\mu\nu}{}^{ab} = -\kappa(\psi_{\mu,\nu}{}^{ab} - \psi_{\nu,\mu}{}^{ab})
\tag{20}
$$

Let us give here useful identities:

$$
\begin{aligned}
(D\Sigma)_{\mu\nu}{}^a - D_\mu \Sigma_\nu{}^a + D_\nu \Sigma_\mu{}^a &= \kappa(\Sigma_{\mu,\nu}{}^a - \Sigma_{\nu,\mu}{}^a) \\
(D\Sigma)_\mu - \frac{1}{2} D_\mu \Sigma &= \frac{\kappa}{2} \Sigma_\mu
\end{aligned}
\tag{21}
$$

[1]Here and further on we use simple "first Greek — first Latin" rool for the contractions of indices.

Also we introduce a convenient linear combination:

$$\hat{\Sigma}_{\mu\nu}{}^{ab} = \Sigma_{\mu\nu}{}^{ab} - \frac{1}{4}e_{[\mu}{}^{[a}\Sigma_{\nu]}{}^{b]} + \frac{1}{18}\Sigma e_\mu{}^{[a}e_\nu{}^{b]}$$

One of the reason why we choose this particular coefficients is a simple form of transformations for this object, for example:

$$\delta\Sigma_{\mu\nu}{}^{abc} = e_{[\mu}{}^{[a}\chi_{\nu]}{}^{bc]} \quad \Longrightarrow \quad \delta\hat{\Sigma}_{\mu\nu}{}^{ab} = -D_{[\mu}\chi_{\nu]}{}^{ab} \tag{22}$$

Analogously, we introduce field strength for the Ω field:

$$\Omega_{\mu\nu}{}^{abc} = D_\mu\Omega_\nu{}^{abc} - D_\nu\Omega_\mu{}^{abc}$$

It is also not a truly gauge invariant object

$$\delta\Omega_\mu{}^{abc} = D_\mu\eta^{abc} \quad \Longrightarrow \quad \delta\Omega_\mu{}^{ab} = 2\kappa\eta_\mu{}^{ab} \tag{23}$$

The following identities:

$$\begin{aligned}
(D\Omega)_{\mu\nu}{}^{ab} - D_\mu\Omega_\nu{}^{ab} + D_\nu\Omega_\mu{}^{ab} &= -\kappa(\Omega_{\mu,\nu}{}^{ab} - \Omega_{\nu,\mu}{}^{ab}) + \kappa e_{[\mu}{}^{[a}\Omega_{\nu]}{}^{b]} \\
(D\Omega)_\mu{}^a - \frac{1}{2}D_\mu\Omega^a &= -2\kappa\Omega_\mu{}^a
\end{aligned} \tag{24}$$

will be useful as well as convenient combination:

$$\hat{\Omega}_\mu{}^{ab} = \Omega_\mu{}^{ab} - \frac{1}{6}e_\mu{}^{[a}\Omega^{b]} \tag{25}$$

Now we are ready to construct dual formulation for this model. Our task here to solve equations for $R_{\mu\nu}{}^{ab}$, $\Phi_{\mu\nu}{}^a$ and $h_\mu{}^a$ fields. Let us start from the $\Phi_{\mu\nu}{}^a$ equation. Once again we face that this field enters the Lagrangian only linearly, so it's equation couldn't be solved for the field $\Phi_{\mu\nu}{}^a$ itself. But for the case $a_0 \neq 0$ it can be solved for the $\omega_\mu{}^{ab}$ field giving:

$$\omega_\mu{}^{ab} = \frac{1}{a_0}\left[-\frac{3}{2}\hat{\Omega}_\mu{}^{ab} - 6m\Sigma_\mu{}^{ab} + me_\mu{}^{[a}\Sigma^{b]}\right] \tag{26}$$

Hence both $\Phi_{\mu\nu}{}^a$ as well as $\omega_\mu{}^{ab}$ field drop out from the resulting second order Lagrangian. Now let us turn to the equation for $R_{\mu\nu}{}^{ab}$ and $h_\mu{}^a$ fields. Comparing the trace of $R_{\mu\nu}{}^{ab}$ equation with the $h_\mu{}^a$ one it is not hard to check that they are not independent and satisfy:

$$2m\delta_d{}^b\frac{\delta\mathcal{L}}{\delta R_{ab}{}^{cd}} - a_0\frac{\delta\mathcal{L}}{\delta h_a{}^c} = D_b\frac{\delta\mathcal{L}}{\delta\Phi_{ab}{}^c}$$

which is a simple consequence of $z_\mu{}^a$ gauge invariance. So it is not possible to express both $R_{\mu\nu}{}^{ab}$ and $h_\mu{}^a$ in terms of the other fields simultaneously. We proceed by solving equation for $R_{\mu\nu}{}^{ab}$:

$$R_{\mu\nu}{}^{ab} = -\frac{1}{a_0{}^2}\left[3\hat{\Sigma}_{\mu\nu}{}^{ab} + \frac{3m}{2}(\Omega_{\mu,\nu}{}^{ab} - \Omega_{\nu,\mu}{}^{ab}) + \frac{ma_0}{2}e_{[\mu}{}^{[a}h_{\nu]}{}^{b]}\right] \tag{27}$$

Now we put this expression into the initial first order Lagrangian. In this, $h_\mu{}^a$ field completely decouples (and that serves as a check for rather lengthy calculations). As a final result we

obtain a second order Lagrangian containing two fields: "gauge" field $\Sigma_{\mu\nu}{}^{abc}$ and "Goldstone" field $\Omega_\mu{}^{abc}$:

$$
\begin{aligned}
\frac{a_0{}^2}{9}\mathcal{L}_{II} &= \frac{1}{2}(\hat{\Sigma}_{ab}{}^{cd}\hat{\Sigma}_{cd}{}^{ab} - 4\hat{\Sigma}_a{}^b\hat{\Sigma}_b{}^a + \hat{\Sigma}\hat{\Sigma}) - \frac{1}{2}(\omega^{a,bc}\omega_{b,ac} - \omega^a\omega_a) + \\
&\quad +m(\hat{\Sigma}_{ab}{}^{cd}\Omega_{c,d}{}^{ab} + 2\hat{\Sigma}_a{}^b\Omega_b{}^a) - \frac{\kappa}{4}(\Omega^{a,bcd}\Omega_{c,dab} - \Omega^{ab}\Omega_{ba}) \\
&\quad -\frac{a_0{}^2}{6}(\Sigma^{ab,cde}\Sigma_{cd,eab} - 4\Sigma^{a,bc}\Sigma_{b,ac} + \Sigma^a\Sigma_a)
\end{aligned} \tag{28}
$$

This Lagrangian also has general structure common to all gauge invariant Lagrangians describing massive particles: it contains a sum of the kinetic terms for two fields, cross terms with one derivative and mass terms. And indeed this Lagrangian is invariant under the following gauge transformations:

$$
\delta\Sigma_{\mu\nu}{}^{abc} = D_\mu\psi_\nu{}^{abc} - D_\nu\psi_\mu{}^{abc} \qquad \delta\Omega_\mu{}^{abc} = 4m\psi_\mu{}^{abc} \tag{29}
$$

Bur this time a Goldstone field $\Omega_\mu{}^{abc}$ is a one form being simultaneously a gauge field having it's own gauge invariance:

$$
\delta\Omega_\mu{}^{abc} = D_\mu\eta^{abc} \qquad \delta\Sigma_{\mu\nu}{}^{abc} = 0 \tag{30}
$$

At last, as a remnant of χ invariance of the initial first order Lagrangian, second order Lagrangian is invariant under the local shifts:

$$
\delta\Sigma_{\mu\nu}{}^{abc} = e_{[\mu}{}^{[a}\chi_{\nu]}{}^{bc]} \qquad \delta\Omega_\mu{}^{abc} = 0 \tag{31}
$$

Conclusion

In this paper we have shown that gauge invariant description of massive particles allows one to resolve ambiguities arising in the construction of dual formulations using first order parent Lagrangians. In this, one can easily consider general case with nonzero masses as well as cosmological term. Note that the resulting second order dual Lagrangians also turn out to be gauge invariant. We restrict ourselves by considering two concrete examples, but it's evident that such a procedure could be easily generalized for other cases as well. At the same time, there are interesting models like those describing partially massless particles that requires special treatment and should be considered separately.

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